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Noncommutative integrability on noncompact invariant manifolds

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Abstract

The Mishchenko–Fomenko theorem on noncommutative completely integrable Hamiltonian systems on a symplectic manifold is extended to the case of noncompact invariant submanifolds.

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1. Introduction

We are concerned with the classical theorems on Abelian and noncommutative integrability of Hamiltonian systems on a symplectic manifold. These are the Liouville–Arnold theorem on Abelian completely integrable systems (henceforth CIS) [1, 2, 26], the Poincaré–Lyapounov–Nekhoroshev theorem on Abelian partially integrable systems [17, 31, 32] and the Mishchenko–Fomenko one on nocommutative CISs [14, 25, 30]. These theorems state the existence of (generalized) action–angle coordinates around a compact invariant submanifold, which is a torus. However, there is a topological obstruction to the existence of global action–angle coordinates [10, 11]. The Liouville–Arnold and Nekhoroshev theorems have been extended to noncompact invariant submanifolds, which are toroidal cylinders [15, 16, 22, 38]. In particular, this is the case of time-dependent CISs [21, 23]. Any time-dependent CIS of m degrees of freedom can be represented as an autonomous one of $m + 1$ degrees of freedom on a homogeneous momentum phase space, where time is a generalized angle coordinate. Therefore, we further consider autonomous CISs.

Our goal here is the following generalization of the Mishchenko–Fomenko theorem to noncommutative CISs whose invariant submanifolds need not be compact.

Theorem 1. *Let* (Z, Ω) *be a connected symplectic* $2n$*-dimensional real smooth manifold and* $(C^\infty(Z), \{,\})$ *the Poisson algebra of smooth real functions on* Z*. Let a subset* $H = (H_1, \dots, H_k)$*,* $n \leqslant k < 2n$*, of* $C^\infty(Z)$ *obey the following conditions.*

(i) The Hamiltonian vector fields ϑ_i *of functions* H_i *are complete.*

(ii) *The map* $H : Z \to \mathbb{R}^k$ *is a submersion with connected and mutually diffeomorphic fibres, i.e.,*

$$H : Z \to N = H(Z) \tag{1}$$

is a fibred manifold over a connected open subset $N \subset \mathbb{R}^k$.

(iii) *There exist real smooth functions* $s_{ij} : N \to \mathbb{R}$ *such that*

$$\{H_i, H_j\} = s_{ij} \circ H, \qquad i, j = 1, \ldots, k. \tag{2}$$

(iv) *The matrix function with the entries* s_{ij} *(2) is of constant co-rank* $m = 2n - k$ *at all points of* N.

Then the following hold.

(I) *The fibres of* H *(1) are diffeomorphic to a toroidal cylinder*

$$\mathbb{R}^{m-r} \times T^r. \tag{3}$$

(II) *Given a fibre* M *of* H *(1), there exists an open saturated neighbourhood* U_M *of it (i.e., a fibre through a point of* U_M *belongs to* U_M*), which is a trivial principal bundle with the structure group (3).*

(III) *Given standard coordinates* (y^λ) *on the toroidal cylinder (3), the neighbourhood* U_M *is provided with bundle coordinates* $(J_\lambda, p_A, q^A, y^\lambda)$*, called the generalized action–angle coordinates, which are the Darboux coordinates of the symplectic form* Ω *on* U_M*, i.e.,*

$$\Omega = \mathrm{d}J_\lambda \wedge \mathrm{d}y^\lambda + \mathrm{d}p_A \wedge \mathrm{d}q^A. \tag{4}$$

In Hamiltonian mechanics, one can think of functions H_i in theorem 1 as being the integrals of motion of a CIS. Their level surfaces (fibres of H) are invariant submanifolds of a CIS.

2. Abelian completely and partially integrable systems

The proof of theorem 1 is based on the fact that an invariant submanifold of a noncommutative CIS is a maximal integral manifold of some Abelian partially integrable Hamiltonian system [14].

If $k = n$, theorem 1 provides the above-mentioned extension of the Liouville–Arnold theorem to Abelian CISs whose invariant submanifolds are noncompact ([38, theorem 6.1], [15, theorem 1]). Note that the proof of theorem 6.1 [38] differs from that of theorem 1 [15]. It is based on lemma 6.4. The statement of its corollary 6.3 is equivalent to the assumption of lemmas 6.1–6.4 that an imbedded invariant submanifold $N_x \subset K$ admits a Lagrangian transversal submanifold $W \subset K$ through x. Apparently, one can avoid the construction of $T^*(W)$ from the proof and, instead of the map γ, consider the map

$$W \times \mathbb{R}^n \longrightarrow (W \times \mathbb{R}^n)/\mathbb{Z}^{m(x)} \longrightarrow \alpha(W \times \mathbb{R}^n).$$

One also need not appeal to the concept of many-valued functions φ_i, but can show that the fibred manifold $\alpha(W \times \mathbb{R}^n) \to W$ is a fibre bundle. This is always true if its fibres are tori and, if $m(x) < n$, follows from the fact that sections l_i of $W \times \mathbb{R}^n \to W$ introduced in lemma 6.2 are smooth.

Condition (ii) of theorem 1 implies that the functions $\{H_\lambda\}$ are independent on Z, i.e., the n-form $\overset{n}{\wedge}\, \mathrm{d}H_\lambda$ nowhere vanishes. Accordingly, the Hamiltonian vector fields ϑ_λ of these functions are independent on Z, i.e., the multivector field $\overset{n}{\wedge}\, \vartheta_\lambda$ nowhere vanishes. If $k = n$, these vector fields are mutually commutative, and they span a regular involutive n-dimensional distribution on Z whose maximal integral manifolds are exactly fibres of the fibred manifold (1). Thus, every fibre of H (1) admits n-independent complete vector fields, i.e., it is a locally affine manifold and, consequently, diffeomorphic to a toroidal cylinder.

Considering an Abelian CIS around some compact invariant submanifold, we come to the Liouville–Arnold theorem (somebody also calls it the Liouville–Mineur–Arnold theorem [39]). Instead of conditions (i) and (ii) of theorem 1, one can suppose that the integrals of motion $\{H_\lambda\}$ are independent almost everywhere on a symplectic manifold Z, i.e., the set of points where the exterior form $\overset{n}{\wedge}\, \mathrm{d}H_\lambda$ (or, equivalently, the multivector field $\overset{n}{\wedge}\, \vartheta_\lambda$) vanishes is nowhere dense. In this case, connected components of level surfaces of functions $\{H_\lambda\}$ form a singular Stefan foliation $\mathcal{F}$ of Z whose leaves are both the maximal integral manifolds of the singular involutive distribution spanned by the vector fields ϑ_λ and the orbits of the pseudogroup G of local diffeomorphisms of Z generated by flows of these vector fields [36, 37]. Let M be a leaf of $\mathcal{F}$ through a regular point $z \in Z$, where $\overset{n}{\wedge}\, \vartheta_\lambda \neq 0$. It is regular everywhere because the group G preserves $\overset{n}{\wedge}\, \vartheta_\lambda$. If M is compact and connected, there exists its saturated open neighbourhood U_M such that the map H restricted to U_M satisfies condition (ii) of theorem 1, i.e., the foliation $\mathcal{F}$ of U_M is a fibred manifold in tori T^n. Since its fibres are compact, U_M is a bundle [29]. Hence, it contains a saturated open neighbourhood of M, say again U_M, which is a trivial principal bundle with the structure group T^n. Providing U_M with the Darboux (action–angle) coordinates $(J_\lambda, \alpha^\lambda)$, one uses the fact that there are no linear functions on a torus T^n.

The Poincaré–Lyapounov–Nekhoroshev theorem generalizes the Liouville–Arnold one to partially integrable systems characterized by $k < n$ independent integrals of motion H_λ in involution. In this case, one deals with k-dimensional maximal integral manifolds of the distribution spanned by Hamiltonian vector fields ϑ_λ of the integrals of motion H_λ. The Poincaré–Lyapounov–Nekhoroshev theorem imposes a sufficient condition which Hamiltonian vector fields ϑ_λ must satisfy in order that their compact maximal integral manifold M admits an open neighbourhood fibred in tori [17, 18]. Such a condition has been also investigated in the case of noncommutative vector fields depending on parameters [19]. Extending the Poincaré–Lyapounov–Nekhoroshev theorem to the case of noncompact integral submanifolds, we in fact assumed from the beginning that these submanifolds form a fibration [15, 22, 23]. In a more general setting, we have studied the property of a given dynamical system to be Hamiltonian relative to different Poisson structures [5, 22, 35]. As is well known, any integrable Hamiltonian system is Hamiltonian relative to different symplectic and Poisson structures, whose variety has been analysed from different viewpoints [3, 4, 8, 12, 13, 28, 34]. One of the reasons is that bi-Hamiltonian systems have a large supply of integrals of motion. Here, we refer to our following result on partially integrable systems on a symplectic manifold ([22, theorem 6]).

Theorem 2. *Given a* $2n$*-dimensional symplectic manifold* (Z, Ω)*, let* $\{H_1, \ldots, H_m\}$*,* $m \leqslant n$*, be smooth real functions on Z in involution which satisfy the following conditions.*

(i) The functions H_λ *are everywhere independent.*
(ii) Their Hamiltonian vector fields ϑ_λ *are complete.*
(iii) These vector fields span a regular distribution whose maximal integral manifolds form a fibration $\mathcal{F}$ *of Z with diffeomorphic fibres.*

Then the following hold.

(I) All fibres of $\mathcal{F}$ *are diffeomorphic to a toroidal cylinder (3).*
(II) There is an open saturated neighbourhood U_M *of every fibre M of* $\mathcal{F}$ *which is a trivial principal bundle with the structure group (3).*

(III) *Given standard coordinates* (y^λ) *on the toroidal cylinder (3), the neighbourhood* U_M *is endowed with the bundle coordinates* $(J_\lambda, p_A, q^A, y^\lambda)$ *such that the symplectic form* Ω *is brought into the form (4).*

Theorem 2 provides the above-mentioned generalization of the Poincaré–Lyapounov–Nekhoroshev theorem to the case of noncompact invariant submanifolds. A geometric aspect of this generalization is the following. Any fibred manifold whose fibres are diffeomorphic either to $\mathbb{R}^r$ or a compact connected manifold K (e.g., a torus) is a fibre bundle [29]. However, a fibred manifold whose fibres are diffeomorphic to a product $\mathbb{R}^r \times K$ (e.g., a toroidal cylinder (3)) need not be a fibre bundle (see [20, example 1.2.2]).

3. The proof of theorem 1

Theorem 2 is the final step of the proof of theorem 1. Condition (iv) of theorem 1 implies that an m-dimensional invariant submanifold of a noncommutative CIS is a maximal integral manifold of some Abelian partially integrable Hamiltonian system obeying the conditions of theorem 2. The proof of this fact is based on the following two assertions [14, 27].

Lemma 3. *Given a symplectic manifold* (Z, Ω)*, let* $H : Z \to N$ *be a fibred manifold such that, for any two functions* f, f' *constant on fibres of* H*, their Poisson bracket* $\{f, f'\}$ *is so. Then* N *is provided with a unique co-induced Poisson structure* $\{,\}_N$ *such that* H *is a Poisson morphism.*

Since any function constant on fibres of H is a pull-back of some function on N, the condition of lemma 3 is satisfied due to item (iii) of theorem 1. Thus, the base N of fibration (1) is endowed with a co-induced Poisson structure.

Lemma 4. *Given a fibred manifold* $H : Z \to N$*, the following conditions are equivalent:*

(i) *the rank of the co-induced Poisson structure* $\{,\}_N$ *on* N *equals* $2\dim \mathrm{N} - \dim \mathrm{Z}$*,*
(ii) *the fibres of* H *are isotropic,*
(iii) *the fibres of* H *are maximal integral manifolds of the involutive distribution spanned by the Hamiltonian vector fields of the pull-back* H^*C *of Casimir functions* C *of the Poisson algebra on* N*.*

It is readily observed that condition (i) of lemma 4 is satisfied due to assumption (iv) of theorem 1. It follows that every fibre M of fibration (1) is a maximal integral manifold of the involutive distribution spanned by the Hamiltonian vector fields υ_λ of the pull-back H^*C_λ of m-independent Casimir functions $\{C_1, \ldots, C_m\}$ on an open neighbourhood N_M of the point $H(M)$. Let us put $U_M = H^{-1}(N_M)$. It is an open saturated neighbourhood of M. Since

$$H^*C_\lambda(z) = (C_\lambda \circ H)(z) = C_\lambda(H_i(z)), \qquad z \in U_M, \tag{5}$$

the Hamiltonian vector fields υ_λ on M are linear combinations of Hamiltonian vector fields ϑ_i of the functions H_i and, therefore, they are complete on M. Similarly, they are complete on any fibre of U_M and, consequently, on U_M. Thus, the conditions of theorem 2 hold on U_M. This completes the proof of theorem 1.

The proof of theorem 1 gives something more. Let $\{H_i\}$ be the integrals of motion of a Hamiltonian $\mathcal{H}$. Since (J_λ, p_A, q^A) are coordinates on N, they are also the integrals of motion of $\mathcal{H}$. Therefore, the Hamiltonian $\mathcal{H}$ depends only on the action coordinates J_λ, and the equation of motion reads

$$\dot{y}^\lambda = \frac{\partial \mathcal{H}}{\partial J_\lambda}, \qquad J_\lambda = \text{const}, \qquad q^A = \text{const}, \qquad p_A = \text{const}.$$

Though the integrals of motion H_i are smooth functions of coordinates (J_λ, q^A, p_A), the Casimir functions

$$C_\lambda(H_i(J_\mu, q^A, p_A)) = C_\lambda(J_\mu)$$

depend only on the action coordinates J_λ. Moreover, a Hamiltonian

$$\mathcal{H}(J_\mu) = \mathcal{H}(C_\lambda(H_i(J_\mu, q^A, p_A))$$

is expressed in the integrals of motion H_i through the Casimir functions (5).

Let us note that, under the assumptions of the Mishchenko–Fomenko theorem, a noncommutative CIS is also integrable in the Abelian sense. Namely, it admits n-independent integrals of motion in involution [6]. Under the conditions of theorem 1, such integrals of motion in involution exist, too. All of them are the pull-back of functions on N. However, one must justify that they obey condition (iii) of theorem 2 in order to characterize an Abelian CIS.

4. Example

The original Mishchenko–Fomenko theorem is restricted to CISs whose integrals of motion $\{H_1, \ldots, H_k\}$ form a k-dimensional real Lie algebra $\mathcal{G}$ of rank m with the commutation relations

$$\{H_i, H_j\} = c_{ij}^h H_h, \qquad c_{ij}^h = \text{const}.$$

In this case, nonvanishing complete Hamiltonian vector fields ϑ_i of H_i define a free Hamiltonian action on Z of some connected Lie group G whose Lie algebra is isomorphic to $\mathcal{G}$. Orbits of G coincide with k-dimensional maximal integral manifolds of the regular distribution on Z spanned by Hamiltonian vector fields ϑ_i [37]. Furthermore, one can treat H (1) as an equivariant momentum mapping of Z to the Lie co-algebra $\mathcal{G}^*$, provided with the coordinates $x_i(H(z)) = H_i(z), z \in Z$, [23, 24]. In this case, the co-induced Poisson structure $\{,\}_N$ in lemma 3 coincides with the canonical Lie–Poisson structure on $\mathcal{G}^*$ given by the Poisson bivector field

$$w = \tfrac{1}{2} c_{ij}^h x_h \partial^i \wedge \partial^j.$$

Recall that the co-adjoint action of $\mathcal{G}$ on $\mathcal{G}^*$ reads $\varepsilon_i(x_j) = c_{ij}^h x_h$, where $\{\varepsilon_i\}$ is a basis for $\mathcal{G}$. Casimir functions of the Lie–Poisson structure are exactly the co-adjoint invariant functions on $\mathcal{G}^*$. They are constant on orbits of the co-adjoint action of G on $\mathcal{G}^*$ which coincide with leaves of the symplectic foliation of $\mathcal{G}^*$. Given a point $z \in Z$ and the orbit G_z of G in Z through z, the fibration H (1) projects this orbit onto the orbit $G_{H(z)}$ of the co-adjoint action of G in $\mathcal{G}^*$ through $H(z)$. Moreover, by virtue of item (iii), lemma 4, the inverse image $H^{-1}(G_{H(z)})$ of $G_{H(z)}$ coincides with the orbit G_z. It follows that any orbit of G in Z is fibred in invariant submanifolds.

The Mishchenko–Fomenko theorem has been mainly applied to CISs whose integrals of motion form a compact Lie algebra. Indeed, the group G generated by flows of the Hamiltonian vector fields is compact, and every orbit of G in Z is compact. Since a fibration of a compact manifold possesses compact fibres, any invariant submanifold of such a noncommutative CIS is compact. Therefore, our theorem 1 essentially extends a class of noncommutative CISs under investigation.

For instance, a spherical top exemplifies a noncommutative CIS whose integrals of motion make up the compact Lie algebra $so(3)$ with respect to some symplectic structure.

Let us consider a CIS with the Lie algebra $\mathcal{G} = so(2, 1)$ of the integrals of motion $\{H_1, H_2, H_3\}$ on a four-dimensional symplectic manifold (Z, Ω), namely,

$$\{H_1, H_2\} = -H_3, \qquad \{H_2, H_3\} = H_1, \qquad \{H_3, H_1\} = H_2. \tag{6}$$

The rank of this algebra (the dimension of its Cartan subalgebra) equals 1. Therefore, an invariant submanifold in theorem 1 is $M = \mathbb{R}$, provided with a Cartesian coordinate y. Let us consider its open saturated neighbourhood U_M projected via $H : U_M \to N$ onto a domain $N \subset \mathcal{G}^*$ in the Lie co-algebra $\mathcal{G}^*$ centred at a point $H(M) \in \mathcal{G}^*$ which belongs to an orbit of the co-adjoint action of maximal dimension 2. A domain N is endowed with the coordinates (x_1, x_2, x_3) such that the integrals of motion $\{H_1, H_2, H_3\}$ on $U_M = N \times \mathbb{R}$, coordinated by (x_1, x_2, x_3, y), read

$$H_1 = x_1, \qquad H_2 = x_2, \qquad H_3 = x_3.$$

As was mentioned above, the co-induced Poisson structure on N is the Lie–Poisson structure

$$w = x_2 \partial^3 \wedge \partial^1 - x_3 \partial^1 \wedge \partial^2 + x_1 \partial^2 \wedge \partial^3. \tag{7}$$

Let us endow N with different coordinates (r, x_1, γ) given by the equalities

$$r = \left(x_1^2 + x_2^2 - x_3^2\right)^{1/2}, \qquad x_2 = \left(r^2 - x_1^2\right)^{1/2} \operatorname{ch}\gamma, \qquad x_3 = \left(r^2 - x_1^2\right)^{1/2} \operatorname{sh}\gamma, \tag{8}$$

where r is a Casimir function on $\mathcal{G}^*$. It is readily observed that the coordinates (8) are the Darboux coordinates of the Lie–Poisson structure (7), namely,

$$w = \frac{\partial}{\partial\gamma} \wedge \frac{\partial}{\partial x_1}. \tag{9}$$

Let ϑ_r be the Hamiltonian vector field of the Casimir function r (8). This vector field is a combination

$$\vartheta_r = \frac{1}{r}(x_1\vartheta_1 + x_2\vartheta_2 - x_3\vartheta_3)$$

of the Hamiltonian vector fields ϑ_i of the integrals of motion H_i. Its flows are invariant submanifolds. Let y be a parameter along the flows of this vector field, i.e.,

$$\vartheta_r = \frac{\partial}{\partial y}.$$

Then the Poisson bivector associated with the symplectic form Ω on U_M is

$$W = \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial\gamma} \wedge \frac{\partial}{\partial x_1}. \tag{10}$$

Accordingly, Hamiltonian vector fields of integrals of motion take the form

$$\vartheta_1 = -\frac{\partial}{\partial\gamma},$$

$$\vartheta_2 = r\left(r^2 - x_1^2\right)^{-1/2} \operatorname{ch}\gamma \frac{\partial}{\partial y} + x_1\left(r^2 - x_1^2\right)^{-1/2} \operatorname{ch}\gamma \frac{\partial}{\partial\gamma} + \left(r^2 - x_1^2\right)^{1/2} \operatorname{sh}\gamma \frac{\partial}{\partial x_1},$$

$$\vartheta_3 = r\left(r^2 - x_1^2\right)^{-1/2} \operatorname{sh}\gamma \frac{\partial}{\partial y} + x_1\left(r^2 - x_1^2\right)^{-1/2} \operatorname{sh}\gamma \frac{\partial}{\partial\gamma} + \left(r^2 - x_1^2\right)^{1/2} \operatorname{ch}\gamma \frac{\partial}{\partial x_1}.$$

Thus, a symplectic annulus (U_M, W) around an invariant submanifold $M = \mathbb{R}$ is endowed with the generalized action–angle coordinates (r, x_1, γ, y) and possesses the corresponding noncommutative CIS $\{r, H_1, \gamma\}$ with the commutation relations

$$\{r, H_1\} = 0, \qquad \{r, \gamma\} = 0, \qquad \{H_1, \gamma\} = 1.$$

This CIS is related to the original one by the transformations

$$r = \left(H_1^2 + H_2^2 - H_3^2\right)^{1/2}, \qquad H_2 = \left(r^2 - H_1^2\right)^{1/2} \operatorname{ch} \gamma, \qquad H_3 = \left(r^2 - H_1^2\right)^{1/2} \operatorname{ch} \gamma.$$

Its Hamiltonian is expressed only in the action variable r.

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